Spherical Expansions into Vacuum: A Higher-Order Analysis

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This paper deals with a higher order analysis of the steady spherically symmetric expansion of a monatomic gas into a vacuum. The analysis is based on the B.G.K. model of the Boltzmann equation for a gas having a viscosity-temperature dependence of the form T^{ω}. It is assumed that the gas is collision dominated at a reference radius, so that the reference Knudsen number is small. A near-equilibrium solution is obtained correct to $0(A^{-1})$ where A is inversely proportional to the reference Knudsen number, and is therefore large. The breakdown of this asymptotic inner solution leads to a consideration of the outer nonequilibrium region. Solutions in this outer region have been given previously, and this paper describes the perturbations to these solutions. It is shown that in the outer region perturbations of order A^{-1} and $A^{-4/\alpha}$ exist where $\alpha = 3 + 4(1 - \omega)$. The $0(A^{-1})$ terms can be included in the zeroth-order solution by means of a slight straining of the radial coordinate. Equations for the $0(A^{-4/\alpha})$ perturbations are derived, and asymptotic solutions given. For Maxwell molecules, i.e., $\omega = 1$, these equations can be solved explicitly, and numerical results are presented. It is shown that $\omega = \frac{3}{4}$ is a special case where the first perturbation to temperature in the outer region is of order $\log A/A$. It is also shown that for $\omega < \frac{7}{4}$, a definite nonequilibrium region always exists, whereas for $\omega > \frac{7}{4}$, no breakdown of the inner solution occurs so that the equilibrium solution is uniformly valid. For $\omega = \frac{7}{4}$, the equilibrium solution is rendered uniformly valid using Lighthill's technique of strained coordinates.

Introduction

THE first complete solutions for the steady spherical expansion of a gas from near-equilibrium conditions into a vacuum were given by Edwards and Cheng[1,2] and by Hamel and Willis.[3] These authors truncated moments of Boltzmann's equation using the hypersonic approximation, and showed that the temperature tends to a constant in the far field. These same results were later obtained by Freeman[4] using an asymptotic expansion of the Boltzmann equation for small source Knudsen number. This approach has since been used to study a variety of problems, the details of which can be found in a series of papers by Freeman,[5,6] Grundy,[6,7,8] and Thomas.[5,8]

Freeman's approach is particularly suited to the study of higher approximations, and forms the basis of the higher order analysis of the steady spherical expansion presented in this paper. The analysis is based on the B.G.K. model of the Boltzmann equation for a monatomic gas having a viscosity temperature dependence of the form T^{ω}. It is shown that there exists a perturbation to the temperature in the nonequilibrium region of order A^{-1}, where A^{-1} is proportional to a reference Knudsen number. The solution correct to this order can be included in the zeroth order solution by a slight straining of the radial co-ordinate. There are also terms of order $A^{-4/\alpha}$, where $\alpha = 3 + 4(1 - \omega)$. Equations correct to this order have been obtained, and it is shown that solutions of these equations match the near-equilibrium solution. The heat-transfer terms in the outer region have also been determined and these are found to be of order $A^{-2/\alpha}$.

In the case of Maxwell molecules, i.e., when $\omega = 1$, the temperature corrections in the outer region can be obtained explicitly, and numerical results are presented. The case $\omega = \frac{3}{4}$ is shown to be a special case where the first perturbation to temperature in the outer region is of order $\log A/A$. Similarly $\omega = \frac{7}{4}$ is a special case, and for this value of ω the equilibrium solution can be rendered uniformly valid by means of Lighthill's[11] technique of strained coordinates. It is shown that for $\omega > \frac{7}{4}$ no breakdown occurs, whereas for $\omega < \frac{7}{4}$ a definite nonequilibrium region always exists.

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The author's attention has been drawn to some work by Chen,[9] who has obtained the temperature corrections of order $A^{-4/\alpha}$. He has obtained numerical results for Maxwell molecules using both the B.G.K. model and the full collision integral. However, he has not considered the $0(A^{-1})$ terms, which form the first perturbation when $\omega = 1$. Chen has also simplified the problem by using an ellipsoidal distribution function to estimate the fourth moments. It has been shown by Freeman and Thomas[5] that the fourth moments can be obtained exactly to zeroth order, and that in the case of the B.G.K. model they can be obtained explicitly. It is therefore unnecessary to use an ellipsoidal function, and in this paper no such approximation is employed.

To obtain the order of the perturbation in the outer, nonequilibrium region, a detailed analysis of the inner near-equilibrium region is required. The first part of this paper is therefore concerned with an analysis of the inner region correct to $0(A^{-1})$, i.e., correct to the order of the Navier-Stokes terms.

The Governing Equations

The B.G.K. model of the Boltzmann equation with spherical symmetry can be written in nondimensional form as

$$(\xi + u)\left(\frac{\partial f}{\partial r} - \frac{du}{dr}\frac{\partial f}{\partial \xi}\right) + \frac{\rho^2}{r}\frac{\partial f}{\partial \xi} - \rho\left(\frac{u+\xi}{r}\right)\frac{\partial f}{\partial \rho} = AT^{1-\omega}n(F - f) \quad (1)$$

where ξ and ρ are the radial and perpendicular components of the peculiar molecular velocity, and u is the mean radial velocity. All variables have been made nondimensional with respect to conditions at a radius r_s, where the local Mach number is M_s, the number density n_s, and the temperature T_s. Velocities are normalized with respect to $(RT_s)^{1/2}$, where R is the gas constant. F is the local Maxwellian distribution function based on local conditions, and the parameter A is defined as

$$A = n_s T_s^{1-\omega} r_s \Lambda/(RT_s)^{1/2} \quad (2)$$

where $\Lambda = T_s^{\omega}/\mu_s$, and μ is the coefficient of viscosity. It can be shown that $A \propto K_n^{-1}$, where $K_n = \lambda_s/r_s$. Thus, as $K_n \rightarrow$

0, $A \to \infty$, corresponding to collision dominated conditions near r_s, the reference point.

In terms of the distribution function, the number density and temperature can be expressed in the form

$$n = 2\pi \int_{-\infty}^{+\infty} \int_0^{+\infty} f\rho d\rho d\xi \tag{3a}$$

$$3nT = 2\pi \int_{-\infty}^{+\infty} \int_0^{+\infty} f(\xi^2 + \rho^2)\rho d\rho d\xi \tag{3b}$$

Similarly, the radial and perpendicular components of the pressure tensor are

$$p_{\xi\xi} = n[\xi^2] = 2\pi \int_{-\infty}^{+\infty} \int_0^{+\infty} f\xi^2\rho d\rho d\xi \tag{4a}$$

$$p_{\rho\rho} = n[\rho^2] = 2\pi \int_{-\infty}^{+\infty} \int_0^{+\infty} f\rho^3 d\rho d\xi \tag{4b}$$

and, in general

$$n[\Gamma] = 2\pi \int_{-\infty}^{+\infty} \int_0^{+\infty} f\Gamma(\xi,\rho)\rho d\rho d\xi \tag{5}$$

where $\Gamma(\xi,\rho)$ is any molecular property.

Moment equations can now be obtained from the Boltzmann equation, but these do not, in general, form a closed system. In the limit $A \to \infty$, $r = 0(1)$, a perturbation solution of the B.G.K. equation can be obtained. This has the equilibrium Maxwellian distribution as its first term. From this near-equilibrium result the heat transfer and stress terms can be obtained, namely

$$q = -5A^{-1}T^{\omega}dT/dr + 0(A^{-2}) \tag{6}$$

$$p_{\xi\xi} = p - \tfrac{4}{3} A^{-1}T^{\omega}r[(d/dr)(u/r)] + 0(A^{-2}) \tag{7}$$

where the heat transfer q is defined as

$$q = n([\xi^3] + [\xi\rho^2]) \tag{8}$$

and p is the hydrostatic pressure. With the aid of Eqs. (6) and (7), the moment equations become a closed set, corresponding to the Navier-Stokes equations.

The Near-Equilibrium Solutions

When the hydrodynamic variables are expanded in the form,

$$\begin{aligned} n &= n_0 + A^{-1}n_1 + \dots \\ T &= T_0 + A^{-1}T_1 + \dots \\ u &= u_0 + A^{-1}u_1 + \dots \end{aligned} \tag{9}$$

and substituted into the Navier-Stokes form of the moment equations, equations for the zeroth and first-order terms can be obtained. The zeroth-order equations, corresponding to the inviscid flow equations, can be integrated giving

$$\begin{aligned} n_0u_0 &= (\tfrac{5}{3})^{1/2}M_s/r^2 \\ T_0 &= n_0^{2/3} \\ u_0^2 + 5(\tfrac{5}{3})^{1/2}M_s^{2/3}u_0^{-2/3}r^{-4/3} &= (\tfrac{5}{3})(3 + M_s^2) \end{aligned} \tag{10}$$

The first-order equations become

$$n_0u_1 = -n_1u_0 \tag{11}$$

$$p_1 = n_1T_0 + n_0T_1 \tag{12}$$

$$\frac{dp_1}{dr} + M_s\frac{(\frac{5}{3})^{1/2}}{r^2}\frac{du_1}{dr} = \frac{4}{3r^3}\frac{d}{dr}\left\{r^4T_0^{\omega}\frac{d}{dr}\left(\frac{u_0}{r}\right)\right\} \tag{13}$$

$$\frac{3M_s(\frac{5}{3})^{1/2}}{r^4}\frac{d}{dr}(r^2T_1) + 2p_1r\frac{d}{dr}\left(\frac{u_0}{r}\right) + 2p_0r\frac{d}{dr}\left(\frac{u_1}{r}\right) = \frac{8}{3}T_0^{\omega}r^2\left\{\frac{d}{dr}\left(\frac{u_0}{r}\right)\right\} + \frac{5}{r^2}\frac{d}{dr}\left\{r^2T_0^{\omega}\frac{dT_0}{dr}\right\} \tag{14}$$

with boundary conditions $n_1 = u_1 = p_1 = T_1 = 0$ at $r = 1$.

These equations can be integrated, to give

$$T_1 = P(r) - \tfrac{2}{3}M_s^{2/3}(\tfrac{5}{3})^{1/2}u_1r^{-4/3}/u_0^{5/3} \tag{15}$$

where

$$P(r) = \tfrac{2}{3}u_0^{-2/3}r^{-4/3}M_s^{(2\omega-3)/3}(\tfrac{5}{3})^{(2\omega-3)/6} \times \int_1^r u_0^{2(1-\omega)/3}r^{4(1-\omega)/3} \times Y(r)dr \tag{16}$$

and

$$Y(r) = r^2\left\{\left(\frac{du_0}{dr}\right)^2 - u_0\frac{d^2u_0}{dr^2}\right\} - \frac{(14 - 4\omega)}{3}u_0r\frac{du_0}{dr} + \frac{4}{3}u_0^2 \tag{17}$$

Also,

$$u_1 = \epsilon^{Z(r)}\int_1^r \times \frac{e^{-Z(x)}\left[-\frac{d}{dx}(n_0P) + \frac{4}{3x^3}\frac{d}{dx}\left(x^4T_0^{\omega}\frac{d}{dx}\frac{u_0}{x}\right)\right]dx}{\left[M_s\frac{(\frac{5}{3})^{1/2}}{x^2} - \frac{5}{3}M_s^{5/3}\frac{(\frac{5}{3})^{5/6}}{x^{10/3}u_0^{8/3}}\right]} \tag{18}$$

where

$$Z(r) = \int_1^r \frac{\frac{5}{3}M_s^{5/3}\left(\frac{5}{3}\right)^{5/6}\frac{d}{dx}(x^{-10/3}/u_0^{8/3})}{\left[M_s\frac{(\frac{5}{3})^{1/2}}{x^2} - \frac{5}{3}M_s\frac{(\frac{5}{3})^{5/6}}{x^{10/3}u_0^{8/3}}\right]}dx \tag{19}$$

The location of the sonic point in the expansion is given by

$$r^* = 4M_s^{1/2}/(3 + M_s^2)$$

and as $r \to r^*$, u_1 and T_1 become singular. This is the sonic point singularity which has been studied by Grundy.[10] As $M_s \to 1$, $r^* \to 1$, and for the flow in the neighborhood of the reference point ($r = 1$) to be of the near-equilibrium type, M_s must be greater than one.

In the limit $r \to \infty$, $r/A \to 0$, the zeroth- and first-order terms can be expressed in terms of r, namely

$$n_0 \sim M_s(3 + M_s)^{-1/2}/r^2 + \tfrac{3}{2}M_s^{5/3}(3 + M_s^2)^{-11/6}r^{-10/3} + \dots \tag{20a}$$

$$u_0 \sim (\tfrac{5}{3})^{1/2}(3 + M_s^2)^{1/2} - \tfrac{3}{2}M_s^{2/3}(\tfrac{5}{3})^{1/2}(3 + M_s^2)^{-5/6}r^{-4/3} + \dots \tag{20b}$$

$$T_0 \sim M_s^{2/3}(3 + M_s)^{-1/3}/r^{4/3} + M_s^{4/3}r^{-8/3}/(3 + M_s^2)^{5/3} + \dots \tag{20c}$$

$$n_1 \sim \frac{-M_s(\frac{5}{3})^{-1/2}b}{(3 + M_s^2)r^2} + \frac{16(3 - \omega)(\frac{5}{3})^{-1/2}M_s^{2\omega/3}r^{-(4\omega+3)/3}}{3(7 - 4\omega)(3 + M_s^2)^{(3+2\omega)/6}} + \dots \tag{21a}$$

$$u_1 \sim b - \frac{16(3 - \omega)M_s^{(2\omega-3)/3}(3 + M_s^2)^{(3-2\omega)/6}r^{(-4\omega+3)/3}}{3(7 - 4\omega)} + \dots \tag{21b}$$

$$T_1 \sim \frac{8(3 + M_s^2)^{(3-\omega)/3}M_s^{(2\omega-3)/3}(\frac{5}{3})^{1/2}r^{(3-4\omega)/3}}{3(7 - 4\omega)} + cr^{-4/3} - 4(\tfrac{5}{3})^{1/2}\{[(88\omega^2 - 350\omega + 423)(3 + M_s^2)^{-(\omega+1)/3} \times M_s^{(2\omega-1)/3}r^{-(4\omega-1)/3}]/15(7 - 4\omega)(3 - 4\omega)\} + \dots \tag{21c}$$

where b and c are numerically determined functions of M_s. They are plotted in Fig. 1. The full numerical solutions for u_1 and T_1 are shown in Fig. 2.

From the form of the asymptotic expansions of the inner solution for large r, it can be seen that the inner solution will break down when $r = 0(A^{3/\alpha})$ and $\alpha = 3 + 4(1 - \omega)$. This fact was first noted in Refs. 1 and 3. It will be shown in the next section that the parameters b and c introduce terms of order A^{-1} into the outer solution. Because these parameters do not appear in Chen's[9] analysis, the $0(A^{-1})$ terms are absent. It is not clear why these terms are omitted.

The Outer Region

Following Freeman,[4] the variables appropriate to the outer region are

$$s_1 = rA^{-3/\alpha},\ \tau = TA^{4/\alpha},\ N = nA^{6/\alpha}$$
$$u = U,\ \phi = \xi A^{2/\alpha},\ \psi = \rho A^{2/\alpha} \tag{22}$$

In terms of these variables, the B.G.K. equation can be written as

$$\frac{\partial f}{\partial s_1} - \frac{U\psi}{s_1}\frac{\partial f}{\partial \psi} - \frac{dU}{ds_1}\phi\frac{\partial f}{\partial \phi} - A^{2/\alpha}U\frac{dU}{ds_1}\frac{\partial f}{\partial \phi} +$$
$$A^{-2/\alpha}\left[\phi\frac{\partial f}{\partial s_1} + \frac{\psi^2}{s_1}\frac{\partial f}{\partial \phi} - \frac{\phi\psi}{s_1}\frac{\partial f}{\partial \psi}\right] = N\tau^{1-\omega}[F - f] \tag{23}$$

The appropriate moment equations can now be generated. To zeroth order, they form a closed set.[1,3,4] The solutions of these moment equations must match the outer limits of the inner solutions, given by Eqs. (20) and (21), as $s_1 \to 0$. When these limiting solutions are written in outer variables, it appears that the thermodynamic variables and moments in the outer, nonequilibrium, region must be expanded in the form

$$[\Gamma] = [\Gamma]_0 + A^{-1}[\Gamma]_1 + A^{-4/\alpha}[\Gamma]_2 + \ldots \tag{24}$$

However, it is possible to absorb the $0(A^{-1})$ terms in this outer solution by introducing a stretched variable s, given by

$$s_1 = s[M_s(\tfrac{5}{3})^{-1/2}/(3 + M_s^2)]\{1 - (6b/5)(\tfrac{5}{3})^{1/2}(3 + M_s^2)^{-1/2}A^{-1}\} \tag{25}$$

and the variables are then expanded as

$$\tau = \tau_0(s,A^{-1}) + A^{-4/\alpha}\tau_2(s) + \ldots \tag{26a}$$
$$U = U_0(s,A^{-1}) + A^{-4/\alpha}U_2(s) + \ldots \tag{26b}$$
$$N = N_0(s,A^{-1}) + A^{-4/\alpha}N_2(s) + \ldots \tag{26c}$$

The third moments in the inner region can be obtained from the near-equilibrium expansion of the distribution function, e.g., Eq. (6). When these moments are written in outer variables, it is clear that, in the outer region

$$[\phi\psi^2] = A^{-2/\alpha}[\phi\psi^2]_1 + \ldots \tag{27a}$$
$$[\phi^3] = A^{-2/\alpha}[\phi^3]_1 + \ldots \tag{27b}$$

If the expansions given by Eqs. (26) and (27) are now substituted in the moment equations for the outer region, separate equations for the zeroth and $0(A^{-4/\alpha})$ terms can be obtained. Matching conditions in the limit $s \to 0$ are obtained by writing the outer limits of the inner solutions [Eqs. (20) and (21)] in outer variables.

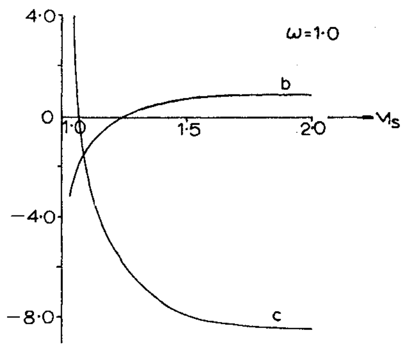

Fig. 1 ***b* and *c* as functions of M_s.**

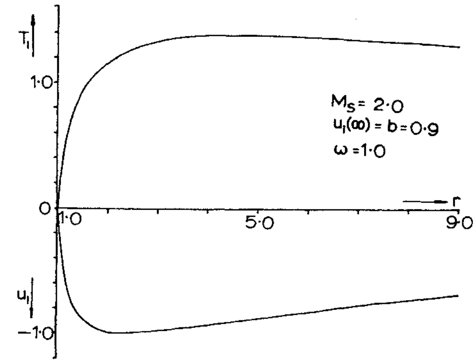

Fig. 2 u_1 **and** T_1 **vs** r.

In what follows, some of the main results are quoted. For a more detailed exposition of the work, the reader is referred to Ref. 13.

The continuity equation yields

$$N_0U_0s_1^2 = M_s(\tfrac{5}{3})^{1/2} \tag{28}$$
$$N_0U_2 = -N_2U_0 \tag{29}$$

and to zeroth order the momentum equation gives U_0 = constant. The matching conditions then yield the solutions

$$U_0 = (\tfrac{5}{3})^{1/2}(3 + M_s^2)^{1/2} + bA^{-1} \tag{30}$$
$$N_0 = [5M_s^{-1}/3(3 + M_s^2)^{5/2}] \times \{1 - 3A^{-1}b(\tfrac{5}{3})^{1/2}/(3 + M_s^2)^{1/2}\}s^{-2} \tag{31}$$

To order $A^{-4/\alpha}$, the momentum equation yields an equation for dU_2/ds which can be integrated to give

$$U_2 = (-3/2\beta)[3\tau_0 + s(d/ds)\tau_0] \text{ where } \beta = (\tfrac{5}{3})^{1/2}(3 + M_s^2)^{1/2} \tag{32}$$

U_2 is plotted as a function of s in Fig. 3. It can be seen that U_2 is everywhere negative.

The energy equation and the $[\phi^2]$ moment equation can be obtained by multiplying the Boltzmann equation, Eq. (23), by $\phi^2 + \psi^2$ and ϕ^2, respectively, and integrating.

These two equations can be combined with the equation of state, $3\tau = [\phi^2] + [\psi^2]$, to produce two equations for τ_0 and τ_2, respectively. These equations are

$$s^2\tau_0'' + s\tau_0'(3 + \tau_0^{1-\omega}/s) + 4\tau_0^{2-\omega}/3s = 0 \tag{33}$$
$$s^2\tau_2'' + s\tau_2'(3 + \tau_0^{1-\omega}/s) + \tau_2\{4\tau_0^{1-\omega}/3s - 2(1 - \omega)s[\phi^2]_0'/3\tau_0\} = 2sV/3 + \tau_0^{1-\omega}R + s(Rs)' = s^2G(s) \tag{34}$$

where

$$V(s) = \frac{-1}{\beta}\left(2(U_2[\phi^2]_0)' + [\phi^3]_1' - \frac{2}{s}[\phi\psi^2]_1\right) \tag{35a}$$

$$\text{and } R(s) = \frac{-1}{3\beta}\left(([\phi^3]_1 + [\phi\psi^2]_1)' + 2U_2'[\phi^2]_0\right) \tag{35b}$$

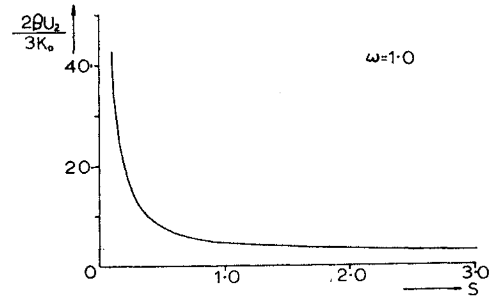

Fig. 3 $-2U_2\beta/3K_0$ **vs** s.

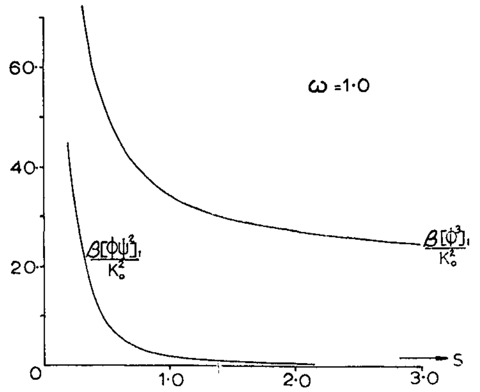

Fig. 4 The third moments vs s.

In addition, it can easily be shown that

$$[\psi^2]_0 = -(3s/2)\tau_0' \text{ and } [\phi^2]_0 = \tfrac{3}{2}(2\tau_0 + s\tau_0') \quad (36)$$

Primes denote differentiation with respect to s. Equation (33) has been studied by Edwards and Cheng[1] and corresponds to the equation obtained by Hamel and Willis.[3] As previously mentioned, the technique of using a scaled Boltzmann equation was developed by Freeman,[4] who rederived Eq. (33) in the form of an integral equation. Equation (33) admits a solution of the form

$$\tau_0 \sim Ks^{-4/3} + 8K^{\omega}s^{(3-4\omega)/3}/3(7-4\omega) + \ldots \quad (37)$$

When this result is matched with the outer limit of the inner solution, it is found that

$$K_0 = \frac{(3+M_s^2)(\frac{5}{3})^{2/3}}{M_s^{2/3}} \quad (38a)$$

$$K_1 = \frac{8b(\frac{5}{3})^{7/6}(3+M_s^2)^{6/7}}{5M_s^{2/3}} + c\,\frac{(3+M_s^2)^{4/3}(\frac{5}{3})^{2/3}}{M_s^{4/3}} \quad (38b)$$

where

$$K = K_0 + A^{-1}K_1 \quad (39)$$

Thus, correct to order A^{-1}, the outer solution for the temperature is $\tau = \tau_0(s,A^{-1})$, where τ_0 is the solution of Eq. (33) subject to the matching conditions given by Eqs. (37) and (38). Before the equation for τ_2 is solved, expressions for the third moments must be obtained. The necessary expressions can be obtained by multiplying the Boltzmann equation [Eq. (23)] by ϕ^3 and $\phi\psi^2$, respectively, and integrating. With the aid of Eq. (27), these moment equations yield two first-order differential equations for $[\phi^3]_1$ and $[\phi\psi^2]_1$ which can be solved explicitly. The results are

$$[\phi^3]_1 = -\beta^{-1}e^{-\gamma(s)}\int_0^s e^{+\gamma(x)}P(x)dx \quad (40)$$

and

$$[\phi\psi^2]_1 = -\beta^{-1}s^{-2}e^{-\gamma(s)}\int_0^s e^{+\gamma(x)}x^2Q(x)dx \quad (41)$$

where

$$\gamma(s) = \int_0^s\left(\frac{\tau_0^{1-\omega}}{x^2} - K_0^{1-\omega}x^{-(10-4\omega)/3} - \frac{8(1-\omega)x^{-1}}{3(7-4\omega)}\right)dx - \frac{3K_0^{1-\omega}s^{(1-4\omega)/3}}{(7-4\omega)} + \frac{8(1-\omega)\log s}{3(7-4\omega)} \quad (42)$$

$$P(s) = \left[3[\phi^2]_0\left(\frac{[\psi^2]_0}{s} - [\phi^2]_0'\right) + [\phi^4]_0' - \frac{3}{s}[\phi^2\psi^2]_0\right] \quad (43a)$$

and

$$Q(s) = \left[[\psi^2]_0\left(\frac{[\psi^2]_0}{s} - [\phi^2]_0'\right) + \frac{1}{s^2}(s^2[\phi^2\psi^2]_0)' - \frac{[\psi^4]_0}{s}\right] \quad (43b)$$

Thus in order to determine P and Q, it is necessary to obtain the fourth moments. Again this can be done by taking the appropriate moments of the Boltzmann equation, and solving the resulting first order differential equations. The required solutions are

$$[\phi^4]_0 = 3e^{-\gamma}\int_0^s e^{\gamma}\tau_0^{3-\omega}\,dx/x^2 \quad (44a)$$

$$[\phi^2\psi^2]_0 = 2s^{-2}e^{-\gamma}\int_0^s e^{\gamma}\tau_0^{3-\omega}\,dx \quad (44b)$$

and

$$[\psi^4]_0 = 8s^{-4}e^{-\gamma}\int_0^s e^{+\gamma}\tau_0^{3-\omega}\,x^2dx \quad (44c)$$

A detailed discussion of the fourth moments can be found in Ref. 5.

Equations (33)–(44) form a numerically integrable set of equations when the boundary conditions are known. These are supplied by the asymptotic solutions in the limit $s \to 0$, and these of course match the outer limits of the inner solutions given by Eqs. (20) and (21).

In this outer region, the heat transfer $H(s)$ is given by

$$H(s) = N_0([\phi^3]_1 + [\phi\psi^2]_1) \quad (45)$$

where

$$H(s) = A^{-(17-3\omega)/\alpha}q \quad (46)$$

Asymptotic Solutions

The important limiting solutions are listed in this section. Again, for a more detailed treatment, see Ref. 13.

As $s \to 0$, the third moments become

$$[\phi^3]_1 \sim \beta^{-1}\left[4K_0^{\omega+1}s^{-(1+4\omega)/3} + 32\,\frac{K_0^{2\omega}(-4\omega^2+8\omega+1)}{3(7-4\omega)}\,s^{(6-8\omega)/3} + \ldots\right] \quad (47)$$

$$[\phi\psi^2]_1 \sim \beta^{-1}\left[8K_0^{\omega+1}s^{-(1+4\omega)/3} - 4K_0^{2\omega}\,\frac{(16\omega^2-128\omega+131)s^{(6-8\omega)/3}}{9(7-4\omega)} + \ldots\right] \quad (48)$$

It can be shown that these solutions match with the inner region. With Eqs. (47) and (48), and the corresponding solutions for the fourth moments, it is possible to obtain the asymptotic form of $G(s)$ and hence to solve for τ_2 as $s \to 0$. The result is

$$\tau_2 \sim \beta^{-2}\left[5K_0^2s^{-8/3} - \frac{4K_0^{\omega+1}(88\omega^2-350\omega+423)}{9(3-4\omega)(7-4\omega)}\,s^{-(1+4\omega)/3} + \ldots\right] \quad (49)$$

Again, this solution matches with the inner region. Further terms can be derived and Eq. (49) can be used to start a numerical solution. It is obvious from Eq. (49) that $\omega = \frac{3}{4}$ and $\omega = \frac{7}{4}$ represent special cases. These are considered in a later section.

It is also instructive to consider the limit $s \to \infty$, i.e., the far field of the nonequilibrium region. In this limit, the equation for τ_0, Eq. (33), admits a solution of the form

$$\tau_0 \sim a_1(1 + 4a_1^{1-\omega}/3s + \ldots) \quad (50)$$

i.e., the temperature approaches a "frozen" limit.[1,3,4] It can also be shown that as $s \to \infty$

$$[\phi^3]_1 \sim (K_0^2/\beta)(a_2 + b_2/s + \ldots)$$
$$\text{and } [\phi\psi^2]_1 \sim (K_0^2/\beta)(b_3/s + \ldots) \quad (51)$$

and that

$$\tau_2 \sim (K_0^2/\beta^2)[a_4 + 0(1/s)] \quad (52)$$

Thus, in the far field, the $0(A^{-4/\alpha})$ perturbation to temperature also tends to a constant. This indicates that the outer asymptotic expansion is well behaved and that no nonuniformities arise.

The Case of Maxwellian Molecules

When $\omega = 1$, Eq. (33) becomes linear and can be solved explicitly. The general solution is

$$\tau_0 = c\psi^*(-\tfrac{4}{3}, -1, 1/s) + (d/s^2)\phi^*(\tfrac{2}{3}, 3, 1/s) \quad (53)$$

where ψ^* and ϕ^* are confluent hypergeometric functions. The solution that matches the inner region as $s \to 0$ is [see Eq. (37)]

$$\tau_0 = K\psi^*(-\tfrac{4}{3}, -1, 1/s) \quad (54)$$

This result was first given in Refs. 1 and 3. Equation (34) is linear, and the left-hand side is identical with Eq. (33). The solution of Eq. (34) which satisfies the matching condition given by Eq. (49) is

$$\tau_2 = \psi^*\left(-\frac{4}{3}, -1, \frac{1}{s}\right)\left\{\left[xe^{-1/x}\phi^*\left(\frac{2}{3}, 3, \frac{1}{x}\right)\bar{G}(x) + \frac{20}{9}x^{-7/3} + \frac{544}{81}x^{-4/3} + \frac{1456}{81}x^{-1/3}\right]dx + \frac{5}{3}s^{-4/3} + \frac{544}{27}s^{-1/3} - \frac{728}{27}s^{2/3}\right\} - \frac{\phi^*}{s^2}\left(\frac{2}{3}, 3, \frac{1}{s}\right)\int_0^s x^3 e^{-1/x}\psi^* \times \left(-\frac{4}{3}, -1, \frac{1}{x}\right)\bar{G}(x)dx \quad (55)$$

where

$$\bar{G}(s) = \Gamma(\tfrac{2}{3})G(s)/\Gamma(3) \quad (56)$$

The third and fourth moments also take on a more tractable form, and satisfy the following expressions:

$$[\phi^3]_1 = -\beta^{-1}e^{1/s}\int_0^s e^{-1/x}P(x)dx \quad (57a)$$

and

$$[\phi^4]_0 = 3K_0^2 e^{1/s}\int_0^s e^{-1/x}\psi^{*2}\left(-\frac{4}{3}, -1, \frac{1}{x}\right)dx/x^2 \quad (57b)$$

with similar expressions for the other third and fourth moments, see Ref. 13. For the case of the full collision integral for Maxwell molecules, the fourth moment equations are coupled, and expressions for the fourth moments cannot be obtained in closed form. Numerical solutions are given in Ref. 5. Similarly, the third moments are coupled, each moment satisfying a second order differential equation. The solution of these equations can however be obtained in closed form, as in Chen's[9] work. The full collision integral has not been considered in this paper, as it makes the analysis more tedious without making any difference in principle. Chen has shown that the numerical results for the B.G.K. and full collision integral models do not differ markedly.

Numerical results for the third moments are presented in Fig. 4. Equation (55) has been numerically evaluated, and the results are shown in Fig. 6. The final "frozen" temperature

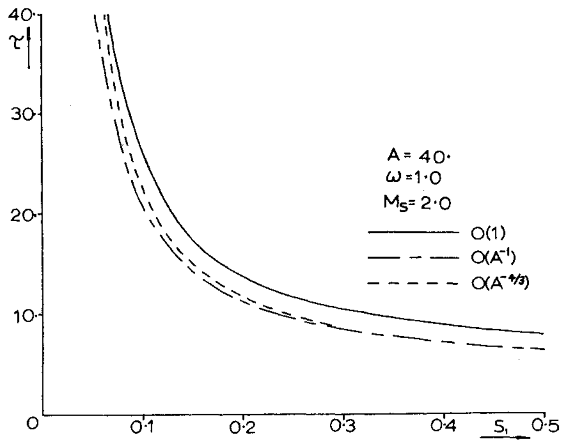

Fig. 5 **τ vs s_1 correct to $0(1), 0(A^{-1})$, and $0(A^{-4/3})$.**

in the outer region becomes, for $\omega = 1$,

$$\tau(\infty) = (K_0 + A^{-1}K_1)\frac{\Gamma(2)}{\Gamma(\frac{2}{3})} + A^{-4/3}\tau_2(\infty) + \ldots \quad (58)$$

Similarly the limiting velocity is given by

$$U(\infty) = \beta + A^{-1}b - A^{-4/3}\frac{9K_0\Gamma(2)}{2\beta\Gamma(\frac{2}{3})} + \ldots \quad (59)$$

Numerical results are presented and discussed in the final section.

The Cases $\omega = \frac{3}{4}$ and $\omega = \frac{7}{4}$

It is obvious from Eq. (49) that solutions when $\omega = \frac{3}{4}$ and $\omega = \frac{7}{4}$ must be considered separately. When $\omega = \frac{3}{4}$, the outer limits of the inner solutions become

$$T_1 \sim 2(\tfrac{5}{3})^{1/2}(3 + M_s^2)^{3/4} \times \left[1 - \frac{7M_s^{2/3}\log r}{(3 + M_s^2)^{4/3}r^{4/3}} + \frac{c_1}{r^{4/3}} + \ldots\right] \quad (60)$$

$$u_1 \sim c_2 + 0(r^{-4/3}) \quad (61)$$

where c_1 and c_2 are constants of integration. Given that T_0 behaves as $r^{-4/3}$ as $r \to \infty$, the inner asymptotic expansion again breaks down, and from Eq. (22) the scaled variables become

$$s_1 = rA^{-3/4}, \qquad \tau = AT \quad (62a)$$
$$N = nA^{3/2}, \qquad u = U \quad (62b)$$

The logarithmic term in Eq. (60) gives rise to a term of order $\log A/A$ in the outer asymptotic expansion, and the appropriate expansion in the nonequilibrium region is thus

$$\tau = \tau_0 + (\log A/A)\tau_1 + A^{-1}\tau_2 + \ldots \quad (63a)$$
$$U = U_0 \qquad + A^{-1}U_2 + \ldots \quad (63b)$$
$$N = N_0 \qquad + A^{-1}N_2 + \ldots \quad (63c)$$

Solutions for U_0, N_0 and τ_0 are again given by Eqs. (30), (31) and (33), except for the omission of the $0(A^{-1})$ terms. The first perturbation to the temperature is given by the homogeneous form of Eq. (34), i.e.,

$$s^2\frac{d^2\tau_1}{ds^2} + s\frac{d\tau_1}{ds}\left(3 + \frac{\tau_0^{1/4}}{s}\right) + \tau_1\left(\frac{4}{3}\frac{\tau_0^{1/4}}{s} - \frac{s}{6\tau_0}\frac{d}{ds}[\phi^2]_0\right) = 0 \quad (64)$$

The matching condition as $s \to 0$ can be obtained by writing Eq. (60) in outer variables. Thus

$$\tau_1 \sim -35(3 + M_s^2)^{3/4}s^{-4/3}/6M_s^{7/6} \text{ as } s \to 0 \quad (65)$$

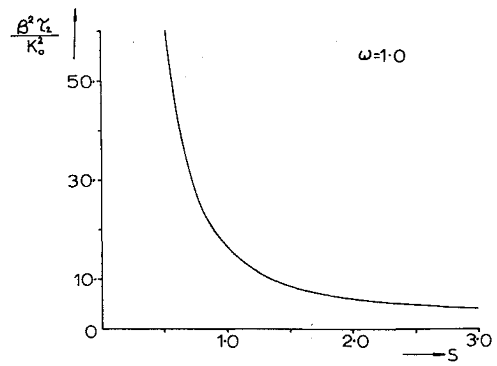

Fig. 6 $\beta^2\tau_2/K_0^2$ **vs** s.

Again, s is given by Eq. (25) minus the $0(A^{-1})$ terms. When $\omega < \frac{3}{4}$, terms of order $A^{-4/\alpha}$ dominate, and when $\omega > \frac{3}{4}$ the dominant perturbation is provided by the $O(A^{-1})$ terms. When $\omega = \frac{3}{4}$, both these perturbations are of the same order and can be combined to give equations and matching conditions for τ_2, N_2 and U_2. Detailed results are not given.

When $\omega = \frac{7}{4}$, it can be shown that as $r \to \infty$

$$T_1 \sim 40M_s^{1/6}(3 + M_s^2)^{5/12} \log r/27(\tfrac{5}{3})^{1/2}r^{4/3} + c_3/r^{4/3} \quad (66)$$

where c_3 is a constant of integration. Since T_0 behaves like $r^{-4/3}$, it is clear that the near-equilibrium expansion will only break down at exponentially large distances downstream. Rather than seeking a uniformly valid solution using matched asymptotic expansions, the form of Eq. (66) suggests the use of Lighthill's[11] technique of strained coordinates. The variable r is replaced by t in the following way

$$r = t + A^{-1}g(t) + \ldots \quad (67)$$

and $g(t)$ is chosen so that T_1 is not more singular than T_0. In fact when

$$g(t) = t[4(\tfrac{5}{3})^{1/2}(3 + M_s^2)^{3/4}/15M_s^{1/2}] \quad (68)$$

the logarithmic term in Eq. (66) does not appear and the resulting near-equilibrium expansion is rendered uniformly valid, i.e., as $t \to \infty$

$$T \sim [M_s^{2/3}t^{-4/3}/(3 + M_s^2)^{1/3} + \ldots] + A^{-1}(c_4t^{-4/3} + \ldots) + \ldots \quad (69)$$

where c_4 is another constant of integration.

Similar expressions can easily be found for velocity and density. It is interesting to note than when $\omega > \frac{7}{4}$, no breakdown of the inner solution occurs, and the near-equilibrium solution in terms of r is uniformly valid. For $\omega < \frac{7}{4}$, a definite nonequilibrium region exists.

Results and Conclusions

In order to minimize the computation, examples have been restricted to the case $\omega = 1$. The results obtained for the higher order inner analysis are presented in Figs. 1 and 2. In particular, the results of Fig. 2 allow the $0(A^{-1})$ perturbation to temperature in the nonequilibrium region to be calculated. In Fig. 5, the temperature correct to $0(A^{-1})$ can be compared with the zeroth-order result when $A^{-1} = 0.025$ and $M_s = 2$. The correction is obviously significant.

Figure 4 shows the variation of both third moments with s. As predicted in Eq. (51), $[\phi^3]_1$ tends to a constant as $s \to \infty$, and $[\phi\psi^2]_1$ tends to zero. The constants appearing in Eq. (51) are

$$a_2 = 20.0; \qquad b_2 = 13.5; \qquad b_3 = 0.30 \quad (70)$$

Thus, with aid of Eqs. (31) and (45), the radial heat transfer in the far field of the nonequilibrium region becomes

$$H(s) \sim 20(3 + M_s^2)(\tfrac{5}{3})^{5/6}/M_s^{1/3}s^2 + 0(1/s^3) \quad (71)$$

The behavior of τ_2 with s is shown in Fig. 6. It was earlier predicted that τ_2 would tend to a constant as $s \to \infty$. The constant appearing in Eq. (52) is given by

$$a_4 = 0.7 \quad (72)$$

In addition to plots of temperature against s_1 correct to $0(1)$ and $0(A^{-1})$, Fig. 5 gives the temperature in the nonequilibrium region correct to $0(A^{-4/3})$. It can be seen that as s_1 increases, temperature perturbations of this order become much smaller than those of order A^{-1}. As $s_1 \to \infty$, the limiting values of temperature and velocity are given by Eqs. (58) and (59). From the numerical solutions these become

$$\tau(\infty) = 4.58 - 33.6A^{-1} + 2.30A^{-4/3} + \ldots \quad (73a)$$

$$U(\infty) = 3.42 + 0.97A^{-1} - 6.02A^{-4/3} + \ldots \quad (73b)$$

For both these results $M_s = 2$. $N(\infty)$ is of course zero. In the previous section, it was pointed out that for $\omega > \frac{3}{4}$, the $0(A^{-1})$ terms provide the dominant perturbation in the outer region. The aforementioned results for the case $\omega = 1$ suggest that sufficient accuracy may be obtained by ignoring the $0(A^{-4/\alpha})$ perturbation to temperature, only including terms of this order when calculating the velocity. This would be a great simplification because it is an easy matter to find U_2 from Eq. (32) when τ_0 is known. For $\omega < \frac{3}{4}$ the $0(A^{-4/\alpha})$ terms dominate and if higher order results are desired, there is no way of avoiding the calculation of τ_2.

The cases $\omega = \frac{3}{4}$ and $\omega = \frac{7}{4}$ have been analyzed in the previous section. The main points of interest are that the first perturbation to temperature in the outer region is of order $\log A/A$ when $\omega = \frac{3}{4}$, and that for $\omega \geq \frac{7}{4}$, the equilibrium solution is uniformly valid.

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